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Infinite-range Ising spin glass in a transverse field

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Abstract. The infinite-range Ising spin glass in the presence of a transverse field is considered. Using the generalised Trotter formula and replica methods, the free energy for the system with quenched random bond interactions is evaluated using the static approximation. It is shown that, when the strength of the transverse field equals the largest eigenvalue of the random bond interaction matrix, the spin glass transition is destroyed. We also show that a replica-symmetric spin glass phase is *stable* in a certain region of the temperature-field phase diagram. Finally we present suggestive arguments which indicate that, when the time dependence of the dynamic susceptibility is included, then the stability of the replica-symmetric phase is enhanced.

1. Introduction

In this paper we consider the effects of transverse field on the infinite-range Ising spin glass model, namely the Sherrington-Kirkpatrick (SK) model [1-3]. The effect of the transverse field on the phase diagram is studied within the so-called static approximation. The primary purpose of the present paper is to analyse the stability of the ordered phase in the presence of the transverse field using the same assumptions used to determine the phase diagram.

There have been several previous studies which have investigated quantum effects in spin glasses [4-10]. Bray and Moore [5] showed that, for Heisenberg spins, the spin glass transition is not destroyed by quantum fluctuations. Most of the previous studies have been focused on the quantum Heisenberg model and there have been relatively few studies of the effect of transverse field on Ising spin glasses. Ishii and Yamamoto [7] have used the transverse field as the perturbation term with the zeroth-order term being the TAP Hamiltonian. Using standard perturbation theory they obtained the phase diagram which is only qualitatively correct. This is because the perturbation was truncated to second order. A different approach was used by Usadel and the phase diagram he obtained is in agreement with the present work [10]. In this paper we use the approach based on the discretised path integral approach [9]. As mentioned above, we examine the stability of the replica-symmetric (RS) spin glass in the presence of a transverse field. Our main result is that, unlike the classical SK model, there is a region in the phase diagram where a RS spin glass phase is stable. An announcement of our results was made elsewhere [11].

There are possible experimental applications of the model that were first discussed by Pirc *et al* [6]. In mixed hydrogen-bonded ferroelectrics, $Rb_{1-x}(NH_4)_xH_2PO_4$, it

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has been shown that for $0.22 \leq x \leq 0.75$ a frozen phase with the characteristic of spin glass has been observed rather than the pure ($x = 0$) ferroelectric phase or ($x = 1$) antiferroelectric phase [12-15]. Here the proton tunnels between the two energy minima in the hydrogen bond, which is mimicked by the transverse field. The pseudospin model is used for the ferroelectric-antiferroelectric part. The randomness arises because of the mixture of ferroelectric-antiferroelectric character of the compound.

2. The model

The Hamiltonian for the infinite-range Ising spin glass in the presence of an external transverse field can be written as

$$\begin{aligned} H &= H_0 + H_1 \\ &= -\sum_{i<j} J_{ij}\sigma_i^z\sigma_j^z - \Gamma \sum_{j=1}^{N} \sigma_j^x \end{aligned} \tag{2.1}$$

where σ_i^z and σ_i^x are the Pauli spin matrices. The quenched bond interactions are assumed to be random and the distribution of J_{ij} is taken to be Gaussian:

$$P(J_{ij}) = (N/2\pi J^2)^{1/2} \exp(-J_{ij}^2 N/2J^2). \tag{2.2}$$

The scaling of $J_{ij} \sim N^{-1/2}$ ensures that the free energy behaves as an extensive variable.

For the above model the free energy can be evaluated using the replica trick [16]

$$-\beta F = [\ln Q] = \lim_{n\to 0} ([Q^n] - 1)/n \tag{2.3}$$

where the square brackets indicate average over the disorder. Because H_0 and H_1 do not commute the evaluation of $[Q^n]$ is more difficult than in the case for the SK model. It can, however, be done using the Trotter formula [17, 18]. For a given realisation of bonds the partition function Q can be evaluated using the eigenstates of σ^z:

$$Q = \operatorname{Tr} e^{-\beta H} = \sum_{\mu} \langle \mu | e^{-\beta H} | \mu \rangle \tag{2.4a}$$

with

$$|\mu\rangle = \prod_{i=1}^{N} |\mu_i\rangle$$

and

$$|\mu_i\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad \text{or} \qquad \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \tag{2.4b}$$

Inserting the appropriate resolution of unity, using the Trotter formula $[Q^n]$ can be easily computed. The resulting free energy per spin can be obtained from (2.3):

$$\begin{aligned} -\frac{\beta F}{N} = \lim_{\substack{P\to\infty \\ n\to 0}} \frac{1}{nN} \Bigg\{ \Bigg[\int \prod_{\substack{(\alpha,\beta) \\ t,t'}} \left(\frac{N}{2\pi}\right)^{1/2} \mathrm{d}y_{tt'}^{(\alpha\beta)} \int \prod_{\substack{\alpha \\ t,t'}} \left(\frac{N}{2\pi}\right)^{1/2} \mathrm{d}x_{tt'}^{\alpha\alpha} \Bigg] \\ \times \exp\Bigg[-N \sum_{\alpha t,t'} \frac{1}{2P^2} (x_{tt'}^{\alpha\alpha})^2 + \sum_{(\alpha\beta)} \sum_{t,t'} \frac{1}{2P^2} (y_{tt'}^{(\alpha\beta)})^2 \\ - \ln \operatorname*{Tr}_{\{\mu_\alpha^{(t)}\}} \exp(H_{\text{eff}}) \Bigg] - 1 \Bigg\} \end{aligned} \tag{2.5a}$$

where

$$H_{\text{eff}} = \frac{\beta J}{P^2} \sum_{(\alpha\beta)} \sum_{t,t'} y_{tt'}^{(\alpha\beta)} \mu_\alpha^{(t)} \mu_\beta^{(t')}$$

$$+ \frac{\beta J}{\sqrt{2}\, P^2} \sum_{\alpha} \sum_{tt'} x_{tt'}^{\alpha\alpha} \mu_\alpha^{(t)} \mu_\alpha^{(t')} + \sum_{\alpha t} \omega (C + \mu_\alpha^{(t)} \mu_\alpha^{(t+1)}) \tag{2.5b}$$

with

$$\omega = \tfrac{1}{2} \ln \coth(\beta\Gamma/P) \tag{2.5c}$$

$$C = \frac{1}{2\omega} \ln[\sinh(\beta\Gamma/P) \cosh(\beta\Gamma/P)]. \tag{2.5d}$$

In the thermodynamic limit $N \to \infty$, the integrals in (2.5*a*) can be evaluated by the steepest descent method, which enables us to replace $y_{tt'}^{(\alpha\beta)}$ and $x_{tt'}^{\alpha\alpha}$ by their stationary values. This leads to

$$x_{tt'}^{\alpha\alpha} = \frac{1}{\sqrt{2}} \beta J \langle \mu_\alpha^{(t)} \mu_\alpha^{(t')} \rangle \equiv \frac{1}{\sqrt{2}} \beta J \chi_{tt'}^{\alpha\alpha} \tag{2.6a}$$

$$y_{tt'}^{(\alpha\beta)} = \beta J \langle \mu_\alpha^{(t)} \mu_\beta^{(t')} \rangle \equiv \beta J q_{tt'}^{\alpha\beta}. \tag{2.6b}$$

The average $\langle\ \rangle$ is done with respect to the effective Hamiltonian given in (2.5*b*):

$$\langle A \rangle = \operatorname*{Tr}_{\{\mu_\alpha^{(t)}\}} A \exp(H_{\text{eff}}) / \operatorname*{Tr}_{\{\mu_\alpha^{(t)}\}} \exp(H_{\text{eff}}). \tag{2.7}$$

The set of equations (2.5) and (2.6) are exact for the infinite-range model. We now assume that the order parameters are independent of the replicas, i.e. a replica-symmetric solution is assumed. As for the SK model, it is easily shown that the phase boundary does not depend on this assumption. In addition, we assume that the order parameters are independent of $t - t'$, i.e. they are static. One can show that $q_{tt'}^{\alpha\beta}$ should be independent of time. This is because $q_{tt'}^{\alpha\beta}$ describes the coupling between distinct replicas and, as a consequence of invariance with respect to the translation of time, $q_{tt'}^{\alpha\beta}$ cannot depend on the time labels. For $\chi_{tt'}$ this ansatz is an approximation whose validity we comment on below. We let

$$\chi_{tt'}^{\alpha\alpha} = \chi \tag{2.8a}$$

and

$$q_{tt'}^{(\alpha\beta)} = q. \tag{2.8b}$$

The free energy per site becomes

$$-\beta f = \max \lim_{\substack{P\to\infty \\ n\to 0 \\ N\to\infty}} \frac{1}{N} \left\{ -\frac{n\beta^2 J^2 \chi^2}{4} - \frac{n(n-1)\beta^2 J^2 q^2}{4} \right.$$

$$+ \ln \int \mathrm{D}z_1 \prod_\alpha \int \mathrm{D}z_{2\alpha} \left(\exp \sum_{\alpha,t} \omega C \right) \operatorname*{Tr}_{\{\mu_\alpha^{(t)}\}}$$

$$\left. \times \exp \sum_t [h_\alpha \mu_\alpha^{(t)} + \omega \mu_\alpha^{(t)} \mu_\alpha^{(t+1)}] \right\} \tag{2.9}$$

where

$$Dz_1 = \frac{dz_1}{\sqrt{2\pi}} \exp(-z_1^2/2) \qquad Dz_{2\alpha} = \frac{dz_{2\alpha}}{\sqrt{2\pi}} \exp(-z_{2\alpha}^2/2)$$

and

$$h_\alpha = \frac{\beta J}{P}(\sqrt{q}\, z_1 + \sqrt{\chi - q}\, z_{2\alpha}).$$

Evaluating the trace and taking the limit $P \to \infty$ followed by the limit $n \to 0$ (2.9) becomes

$$-\beta f = -\tfrac{1}{4}\beta^2 J^2(\chi^2 - q^2) + \int Dz_1 \ln \int Dz_2\, 2\cosh \Xi \tag{2.10a}$$

where

$$\Xi = (\beta^2\Gamma^2 + b^2)^{1/2} \tag{2.10b}$$

$$b = \beta J(\sqrt{q}\, z_1 + \sqrt{\chi - q}\, z_2). \tag{2.10c}$$

The appropriate values of χ and q are determined from the stationarity of the free energy:

$$\partial f/\partial \chi = \partial f/\partial q = 0.$$

This yields the following self-consistent equations for χ and q:

$$\chi = \int Dz_1 \frac{\int Dz_2[(b^2/\Xi^2)\cosh \Xi + (\gamma^2\beta^2 J^2/\Xi^3)\sinh \Xi]}{\int Dz_2 \cosh \Xi} \tag{2.11a}$$

$$q = \int Dz_1 \left(\frac{\int Dz_2[(b/\Xi)\sinh \Xi]}{\int Dz_2 \cosh \Xi}\right)^2 \tag{2.11b}$$

with $\gamma = \Gamma/J$. Equation (2.11) can be solved (numerically) to obtain the phase boundary separating the paramagnetic and the spin glass phase as a function of γ. The transition temperature, $T_c(\Gamma)$, is determined by the following equation:

$$\frac{1}{\beta_c J} = \frac{\int Dz_2[(z_2^2\beta_c J/\Xi_c^2)\cosh \Xi_c + (\gamma^2\beta_c^2 J^2/\Xi_c^3)\sinh \Xi_c]}{\int Dz_2 \cosh \Xi_c} \tag{2.12a}$$

where

$$\beta_c = 1/k_B T_c(\Gamma) \tag{2.12b}$$

and $\Xi_c = \beta_c J(\gamma^2 + z_2^2/\beta_c J)^{1/2}$. On the boundary separating the paramagnetic and the spin glass phase, it can be shown that $\chi = 1/\beta_c J$. In figure 1, the critical temperature $T_c(\Gamma)$ is plotted as a function of the strength of the transverse field. The phase diagrams shown in figure 1 has been previously found by Usadel using a different approach. There has been some controversy regarding the field strength Γ where the spin glass transition is destroyed [20]. The presence of the transverse field leads to a reduction in the transition temperature. When $\Gamma = 0$, we find that $T_c = J/k_B$ in agreement with the result for the SK model. Figure 1 also shows that there exists a critical value of the transverse field above which the spin glass transition does not take place at finite temperature, i.e. $T_c(\Gamma_c) = 0$. The value of the critical field Γ_c, or equivalently $\gamma_c = \Gamma_c/J$, can easily be obtained from (3.1*a*). When $T_c \to 0$, (2.12*a*) gives an equation for γ_c:

$$\frac{1}{\gamma_c} + \frac{1}{\gamma_c(\gamma_c - 1)} = 1 \tag{2.13}$$

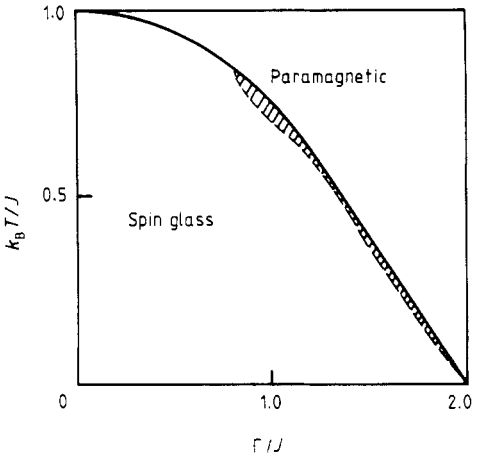

Figure 1. Phase diagram for the infinite-range Ising spin glass in the presence of a transverse field evaluated using replica-symmetric order parameters. In the shaded region of the phase in the spin glass phase, the replica-symmetric solution is stable. The broken line indicates the limit of stability of the replica-symmetric spin glass phase.

which gives $\Gamma_c = 2J$. The value of $2J$ is not a coincidence and it happens to be the largest eigenvalue of the random matrix J_{ij}. It should be emphasised that the inclusion of the time dependence of χ can alter the shape of the phase boundary near $T = 0$ or, more precisely, when $\Gamma \sim 2J$. Thus the vanishing of T_c when $\Gamma = 2J$ is true only in the static approximation.

The entropy of the system can be calculated from the free energy and one finds that

$$\frac{S}{N} = \frac{J^2(\chi^2 - q^2)}{4k_B T^2} + k_B \int \mathrm{D}Z_1 \left(\ln \int \mathrm{D}Z_2 \, 2 \cosh \Xi - \frac{\int \mathrm{D}Z_2 \, \Xi \sinh \Xi}{\int \mathrm{D}Z_2 \cosh \Xi} \right). \tag{2.14}$$

It is instructive to compute the entropy per spin, S/N, near $\gamma_c = 2$ where T_c is small. Equation (2.14) can be expanded in terms of the small parameter $\varepsilon = k_B T/J$ (for $\gamma \sim 2$) and we find that for $\Gamma = 2J + 0^+$

$$S/N = (\tfrac{1}{2} \ln 2 - \tfrac{1}{4}) k_B + \mathrm{O}(\varepsilon^2). \tag{2.15}$$

An important caveat should be added here. Equation (2.15) indicates an extensive ground-state entropy. We expect the entropy to vanish at $T = 0$ and thus we conclude that the static approximation fails in this context. To verify this, we have computed S/N for small T and J/Γ in two ways. For $T = 0$ and $J/\Gamma \ll 1$, (2.14) gives $S/N \sim k_B J^2/4\Gamma^2 + ([J/\Gamma]^4)$. The vanishing of S/N for $J/\Gamma \to 0$ indicates perfect ordering in the transverse direction at $T = 0$. In the limit $J/\Gamma \ll 1$ and $T \to 0$ (which lies entirely in the disordered phase), S/N can also be computed by directly evaluating the partition function. A perturbative expansion in J_{ij} is valid because, for $J/\Gamma \ll 1$, there is no longitudinal ordering. An exact calculation to $\mathrm{O}(J^2/\Gamma^2)$ gives zero ground-state entropy and the leading non-zero terms vanishes exponentially as $T \to 0$. Thus, we are forced to conclude that the static approximation fails when $\Gamma/J \gg 1$. Despite the failure of the static approximation for $T \to 0$, the numerical value of S/N at $T = 0$ (cf (2.15)) is $0.096\, k_B$, which is not very large. Thus the static approximation may be a reasonable first-order approximation.

Given the above, two viewpoints are possible. The first is pessimistic. All of our results hinge on the static approximation. We have just argued that it is incorrect near

$T=0$. This suggests that, for small T, all of our results are suspect. In particular, the region in the ordered phase that we find (cf § 3) supports an RS spin glass phase may be an artefact of the static approximation. The second viewpoint is optimistic. Along the entire phase boundary (not just near $T=0$) we have tried (cf end of § 3) to improve on the static approximation using a perturbation scheme. We find very small changes and the stable RS spin glass region persists. We conclude, however, that our results are at best suggestive and definitive results require further work. It does not appear easy to improve upon the static approximation. In this context, quantum Monte Carlo simulations may prove useful.

3. Stability of the replica-symmetric solution

In this section we analyse the stability of the replica-symmetric solution. The consequences of relaxing the static approximation are examined at the end of this section. When $N\to\infty$, the integrals over the variables $y_{tt'}^{(\alpha\beta)}$ and $\chi_{tt'}^{\alpha\alpha}$ occurring in the expression for free energy (cf (2.5a)) can be evaluated by steepest descent. The result is

$$\frac{\beta F}{N}=-\lim_{P\to\infty}\lim_{n\to 0}\max\frac{1}{N}$$
$$\times\left(-\frac{1}{2P^2}\sum_{tt'}\sum_{(\alpha\beta)}(y_{tt'}^{(\alpha\beta)})^2-\frac{1}{2P^2}\sum_{tt'}\sum_{\alpha}(x_{tt'}^{\alpha\alpha})^2+\ln\operatorname*{Tr}_{\{\mu_\alpha^{(t)}\}}\exp(H_{\text{eff}})\right) \tag{3.1}$$

where H_{eff} is given by (2.5b). In order to assess the stability of the RS spin glass phase in the static approximation, we write

$$x_{tt}^{\alpha\alpha}=x+\varepsilon^{(\alpha)} \tag{3.2a}$$
$$y_{tt'}^{(\alpha\beta)}=y+\eta^{(\alpha\beta)} \tag{3.2b}$$

and expand $\beta F/N$ to second order in the fluctuations $\varepsilon^{(\alpha)}$ and $\eta^{(\alpha\beta)}$. In principle, one should let the fluctuations, $\varepsilon^{(\alpha)}$ and $\eta^{(\alpha\beta)}$, depend on time. To be consistent with the static approximation used in evaluating the free energy we ignore the time dependence. The quantities x and y are the order parameters obtained when $\beta F/N$ (cf (2.5)) is evaluated by steepest descent and are given by

$$x=\beta J\chi/\sqrt{2}\qquad y=\beta Jq. \tag{3.3}$$

The deviation of the free energy $\beta F/N$ from its stationary value is given by

$$\frac{\beta}{N}[F(x+\varepsilon^{(\alpha)},y+\eta^{(\alpha\beta)})-F(x,y)]=-\Delta/2 \tag{3.4a}$$

where

$$\Delta=\sum_{\alpha,\beta}\left[\delta_{\alpha,\beta}-\frac{\beta^2J^2}{P^4}\sum_{t_1,t_2,t_3,t_4}(\langle\mu_\alpha^{(t_1)}\mu_\alpha^{(t_2)}\mu_\beta^{(t_3)}\mu_\beta^{(t_4)}\rangle-\langle\mu_\alpha^{(t_1)}\mu_\alpha^{(t_2)}\rangle\langle\mu_\beta^{(t_3)}\mu_\beta^{(t_4)}\rangle\right]\varepsilon^{(\alpha)}\varepsilon^{(\beta)}$$
$$+\sqrt{2}\frac{\beta^2J^2}{P^4}\sum_{\delta,(\alpha\beta)}\sum_{t_1,\ldots,t_4}[\langle\mu_\delta^{(t_1)}\mu_\delta^{(t_2)}\mu_\alpha^{(t_3)}\mu_\beta^{(t_4)}\rangle-\langle\mu_\delta^{(t_1)}\mu_\delta^{(t_2)}\rangle\langle\mu_\alpha^{(t_3)}\mu_\beta^{(t_4)}\rangle]$$
$$\times\varepsilon^{(\delta)}\eta^{(\alpha\beta)}+\sum_{(\alpha\beta),(\gamma\delta)}\left[\delta_{(\alpha\beta),(\gamma\delta)}-\frac{\beta^2J^2}{P^4}\sum_{t_1,t_2,t_3,t_4}(\langle\mu_\alpha^{(t_1)}\mu_\beta^{(t_2)}\mu_\gamma^{(t_3)}\mu_\delta^{(t_4)}\rangle\right.$$
$$\left.-\langle\mu_\alpha^{(t_1)}\mu_\beta^{(t_2)}\rangle\langle\mu_\gamma^{(t_3)}\mu_\delta^{(t_4)}\rangle)\right]\eta^{(\alpha\beta)}\eta^{(\gamma\delta)}. \tag{3.4b}$$

The average $\langle\ \rangle$ is given by

$$\langle \ldots \rangle = \frac{\mathrm{Tr}_{\{\mu_\alpha^{(t)}\}} \ldots \exp(H_{\mathrm{eff}}^{\mathrm{MF}})}{\mathrm{Tr}_{\{\mu_\alpha^{(t)}\}} \exp(H_{\mathrm{eff}}^{\mathrm{MF}})} \tag{3.5}$$

where $H_{\mathrm{eff}}^{\mathrm{MF}}$ is the same as (2.5) with $x_{tt'}^{\alpha\alpha} = x$ and $y_{tt'}^{\alpha\beta} = y$. The quadratic form Δ should be positive for (3.3) to be a stable solution.

The mathematical structure of Δ is identical to that given by de Almeida and Thouless (AT) [19] in their study of the stability of the SK solution. However, the correlation functions that appear in (3.4*b*) are quite different.

For arbitrary n there are at most five eigenvalues, two of which coincide when the limit $n \to 0$ (which in this problem amounts to setting $n = 0$) is taken [19]. The three eigenvalues are given by

$$\lambda_{1,2} = \tfrac{1}{2}\{(A - B + \bar{P} - 4Q + 3R) \pm [(A - B - \bar{P} + 4Q - 3R)^2 - 8(C - D)^2]^{1/2}\} \tag{3.6a}$$

$$\lambda_3 = \bar{P} - 2Q + R. \tag{3.6b}$$

The variables A, B, $\bar{P}$, Q, R, C, D are the seven matrix elements associated with A [19] and they can be evaluated using the properties of the cyclic one-dimensional Ising chain. Using the results for the correlation functions, and taking the limit of $P \to \infty$, the eigenvalues become

$$\lambda_3 = 1 - \mu \int \mathrm{D}Z_1 (M - N)^2 \tag{3.7a}$$

$$\lambda_2 = 2 - \tfrac{1}{2}\mu \left(\Omega - \int \mathrm{D}Z_1\, M^2\right) - \mu \int \mathrm{D}Z_1 (M - N)(M - 3N) \tag{3.7b}$$

$$\lambda_1 = \left[1 - \tfrac{1}{2}\mu \left(\Omega - \int \mathrm{D}Z_1\, M^2\right)\right]\left(1 - \mu \int \mathrm{D}Z_1 (M - N)(M - 3N)\right) + \mu^2 \left(\Omega - \int \mathrm{D}Z_1\, MN\right)^2 \tag{3.7c}$$

where

$$\mu = \beta^2 J^2 \tag{3.8a}$$

$$M = \frac{\int \mathrm{D}Z_2 [(b^2/\Xi^2) \cosh \Xi + (\gamma^2 \beta^2 J^2/\Xi^3) \sinh \Xi]}{\int \mathrm{D}Z_2 \cosh \Xi} \tag{3.8b}$$

$$N = \left(\frac{\int \mathrm{D}Z_2 [(b/\Xi) \sinh \Xi]}{\int \mathrm{D}Z_2 \cosh \Xi}\right)^2 \tag{3.8c}$$

$$\Omega = \frac{1}{P^4} \sum_{t_1 \ldots t_4} \langle \mu_\alpha^{(t_1)} \mu_\alpha^{(t_2)} \mu_\alpha^{(t_3)} \mu_\alpha^{(t_4)} \rangle. \tag{3.8d}$$

For the stability of the static replica-symmetric solution it is necessary that the eigenvalues λ_1, λ_2, λ_3 be greater than zero. It can be easily shown that, in the paramagnetic phase where $q = 0$, the eigenvalues given in (3.8) are all positive. Thus the paramagnetic phase is stable. In order to establish the stability of the RS solution in the spin glass phase we confine ourselves close to the phase transition boundary, where both q and χ can be expanded as

$$\chi = \frac{1}{\beta_c J} + a_1 t + a_2 t^2 + a_3 t^3 + \ldots \tag{3.9a}$$

$$q = b_1 t + b_2 t^2 + b_3 t^3 + \ldots \tag{3.9b}$$

where

$$t = 1 - T/T_c \qquad \beta_c = 1/k_B T_c. \tag{3.9c}$$

It can be shown by substituting (3.9) into (3.7*b*) and (3.7*c*) that, close to the phase boundary, the eigenvalues λ_1 and λ_2 are greater than zero. Thus the stability of the replica-symmetric state depends on the eigenvalue λ_3.

By substituting (3.9) into the variational equations for q and χ (cf (2.11)) one can obtain the coefficients a_i and b_i. It can be shown that

$$a_1 - b_1 = -1/\beta_c J. \tag{3.10}$$

Expanding M and N in powers of t and using (3.10) it is straightforward to show that the coefficient of the linear t term in the critical eigenvalue λ_3 vanishes for all values of Γ. This is precisely what is found in the SK model. Thus the stability of the RS solution near small values of t is solely determined by the coefficient of the quadratic term. In the SK model, it was shown that the coefficient is negative, implying that the replica-symmetric solution is unstable everywhere in the spin glass phase. In the present problem the coefficient explicitly depends on Γ and one finds that for $\Gamma/J \sim O(1)$ the RS solution is stable. To show this we expand M and N to order t^2:

$$M = \frac{1}{\beta_c J} + t(\alpha_1(\Gamma, T_c) + z_1^2 \alpha_2(\Gamma, T_c)) + t^2 \alpha_3(\Gamma, T_c, z_1) + O(t^3) \tag{3.11a}$$

$$N = t b_1 z_1^2 + \beta_2(\Gamma, T_c, z_1) t^2 \qquad + O(t^3) \tag{3.11b}$$

where the coefficients $\alpha_1, \alpha_2, \alpha_3, \beta_2$ are complicated functions of the indicated arguments. These can be found by straightforward but tedious algebra. One can easily show that

$$\int [\alpha_1(\Gamma, T_c) + z_1^2 \alpha_2(\Gamma, T_c)]\, Dz_1 = a_1 \qquad \int \alpha_3(\Gamma, T_c, z_1)\, Dz_1 = a_2$$
$$\int \beta_2(\Gamma, T_c, z_1)\, Dz_1 = b_2. \tag{3.12}$$

Using (3.11) and (3.12) the stability condition, i.e. the positivity of the critical eigenvalue λ_3, becomes

$$\lambda_3 = 1 - \mu \int (M - N)^2\, Dz_1 = -k(\Gamma) t^2 + O(t^3) > 0 \tag{3.13}$$

where

$$k(\Gamma) = -1 + 2\beta_c J(a_2 - b_2) + \beta_c^2 J^2 [\alpha_1^2 + 2\alpha_1(\alpha_2 - b_1) + 3(\alpha_2 - b_1)^2]. \tag{3.14}$$

We find that

$$k(\Gamma = 0) = \tfrac{4}{3} \tag{3.15}$$

in agreement with the result obtained by de Almeida and Thouless [5]. The RS solution becomes stable when $k(\Gamma) < 0$. The resulting equation for Γ_{LS} is found numerically and it turns out to be $0.85J$. For $\Gamma > \Gamma_{LS}$ we have numerically shown that $k(\Gamma) < 0$ and

consequently the RS solution is stable. The value of $k(\Gamma)$ is found to decrease monotonically from its initial value of $\frac{4}{3}$ as Γ is increased. We have numerically completed the region of the phase diagram where the replica-symmetric solution is stable using the condition given in (3.13). The result is shown as the shaded area in figure 1. It should be noted that $\Gamma_{LS}/J \sim O(1)$, and hence the stability result would be difficult to obtain by perturbative treatment.

The most serious approximation in establishing the stability of the spin glass phase is the neglect of the time dependence of χ. The order parameter χ certainly depends on time and recent Monte Carlo simulations also suggest this [20]. We now present an argument based on a rough perturbative calculation that suggests that the conclusions of the present section should remain valid when one relaxes the static approximation. It is clear that the neglect of the time dependence of χ is valid when $\Gamma < 0.7J$. The precise dependence of T_c for $\Gamma/J \ll 1$ can be shown to be $T_c = J[1-\alpha(\Gamma/J)^2]$ where $\alpha > 0$. In this region the correction to the phase boundary (beyond the static approximation) is proportional to the fourth power of the field. Thus we conclude that the static approximation is valid when Γ/J is small. Recent computer studies suggest, for $\Gamma < 0.70J$, that χ in this region is almost independent of time [20]. For $T_c \sim 0$, the time dependence of χ manifests itself essentially when $t \to 0$ (or $t \to \beta$). To go beyond the static approximation, we use the following ansatz for χ:

$$\chi(t) = \begin{cases} \chi_1 & 0 \leqslant t < \beta/4 \\ \chi_2 & \beta/4 \leqslant t < 3\beta/4 \\ \chi_1 & 3\beta/4 \leqslant t \leqslant \beta. \end{cases} \tag{3.16}$$

The above approximation ensures that χ is symmetric about $t = \beta/2$. With this approximation the effective Hamiltonian can be written as

$$H_{\text{eff}} = H_{\text{eff}}^{\text{MF}} + \delta H \tag{3.17}$$

where $H_{\text{eff}}^{\text{MF}}$ is the effective Hamiltonian within the static approximation. Assuming that δH is small one can show that the stability condition for the critical eigenvalue becomes

$$\lambda_3 = 1 - \mu \int (M-N)^2 \, \mathrm{D}Z_1 - \mu\Delta_1 \geqslant 0 \tag{3.18a}$$

with

$$\Delta_1 = \frac{1}{P^4} \sum_{t_1,t_2,t_3,t_4} \times [\langle \delta H \mu_\alpha^{(t_1)} \mu_\alpha^{(t_2)} \mu_\beta^{(t_3)} \mu_\beta^{(t_4)} \rangle + \langle \delta H \mu_\alpha^{(t_1)} \mu_\beta^{(t_2)} \mu_\gamma^{(t_3)} \mu_\delta^{(t_4)} \rangle - 2\langle \delta H \mu_\alpha^{(t_1)} \mu_\alpha^{(t_2)} \mu_\gamma^{(t_3)} \mu_\delta^{(t_4)} \rangle]. \tag{3.18b}$$

The averages in (3.18*b*) are with respect to $H_{\text{eff}}^{\text{MF}}$. The correlation functions appearing in (3.18*b*) can be calculated using the transfer matrix for a one-dimensional nearest-neighbour Ising model in an external field. We find that Δ_1 is always negative near the phase boundary. This suggests that schemes (3.16) which go beyond the static approximation render the replica-symmetric spin glass phase even more stable. A complete proof of this would require a computation of the time dependence of $\chi(t)$ and the stability analysis around the resulting function. Nevertheless, our argument

suggests that a region of the phase diagram is rendered stable in the presence of a transverse field.

4. Conclusions

The infinite-range Ising spin glass in the presence of an external transverse field provides a natural generalisation of the SK model that accounts for the effects of quenched random interactions on the tunnelling degrees of freedom. Such a model has been argued to be relevant for interpreting the formation of various phases in certain dipolar glassy systems. However, our interest has been to focus primarily on the effect of quantum fluctuations on the stability of the glassy phase. Our study has led us to the following major conclusions.

(i) The spin glass transition temperature T_c is reduced in the presence of a transverse field. It is clear that T_c is a decreasing function of Γ and we have shown that T_c goes to zero when Γ becomes equal to the largest eigenvalue of the random matrix. It is interesting that, in a recent experiment, it has been shown that, when the mixed crystal $Rb_{1-x}(NH_4)_xH_2PO_4$ is subject to external pressure (which is argued to be proportional to Γ), the proton glass transition temperature is found to decrease [9, 13]. At a critical value of the external pressure the transition temperature goes to zero. Although the nature of the ordered phase in finite dimensions cannot be adequately described by infinite-range models, these models are useful in predicting the phase boundary. Consequently the results of our calculations are consistent with the gross features of the above experiments. We have shown that the replica-symmetric solution becomes stable when $\Gamma \sim 0.85J$. Although this important conclusion has been proved only under the static approximation, we have argued that a fuller treatment of the time dependence of χ may lead to a stronger stability of the spin glass phase.

(ii) Recent scaling theories of the ordered phase of the three-dimensional short-range spin glass in zero field suggest the existence of only two pure states. This is in direct contradiction with the Parisi solution to the infinite-range spin glass in the absence of a transverse field [21, 22]. The stability of the replica-symmetric solution of the infinite-range spin glass in the presence of a transverse field suggests the overlap distribution function:

$$P(q) = \left(\sum_{\alpha,\beta} \omega_\alpha \omega_\beta \delta(q - q_{\alpha\beta}) \right) = \tfrac{1}{2}[\delta(q - q(T)) + \delta(q + q(T))] \tag{4.1}$$

where α, β denote pure phases and ω_α is the weight associated with the pure state α. A direct verification of (5.1) can be made by showing that connected correlation functions vanish when the replica-symmetric solution is stable. Notice that within the approximation scheme we have used, (4.1) is only applicable in the shaded region of 1.

(iii) A plausible physical reason for the stability of the replica-symmetric solution can be given in terms of the pure state picture that is suggested by the Parisi solution to the SK model. The ordered phase of the SK model is thought to consist of many pure states, all with the same free energy per spin. The pure states are separated by barrier and the timescale for crossing the barrier scales as $\tau \sim \exp(N^{0.25})$ and hence becomes infinite as $N \to \infty$ [23]. The transverse field essentially induces tunnelling between the pure states and we suggest that the diverging barriers are somehow renormalised to finite values as Γ becomes large enough, i.e. when $\Gamma/J \sim O(1)$. Under

these circumstances, the systems coherently tunnels between what were originally 'pure' states and this leads to the overlap distribution given by (4.1). This hypothesis can be verified by quantum Monte Carlo simulations.

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